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Rotating black holes in a Gödel universe

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Abstract: We construct a five-dimensional, asymptotically Gödel, three-charge black hole *via* dimensional reduction of an asymptotically plane wave, rotating D1-D5-brane solution of type-IIB supergravity. This latter is itself constructed *via* the solution generating procedure of Garfinkle and Vachaspati, applied to the standard rotating D1-D5-brane solution. Taking all charges to be equal gives a "BMPV Gödel black hole", which is closely related to that recently found by Herdeiro. We emphasise, however, the importance of our ten-dimensional microscopic description in terms of branes. We discuss various properties of the asymptotically Gödel black hole, including the physical bound on the rotation of the hole, the existence of closed timelike curves, and possible holographic protection of chronology.

Keywords: D-branes, Black Holes in String Theory, Penrose limit and pp-wave background.

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Contents

1. Introduction

There are many conceptual questions in general relativity which are not expected to be answered within the classical theory itself. A typical example is that of classical solutions with closed timelike curves (CTCs), perhaps the most famous of which is the four-dimensional Gödel universe [1] and various generalisations of it. Such solutions have CTCs for all times, so it is unclear as to what extent the arguments of Hawking [2] concerning chronology protection – themselves quantum mechanical in nature — can be applied.

Much of the recent interest in such Gödel-like solutions is due to the work of [3], where it was shown that supersymmetric generalisations of the four-dimensional Gödel universe are solutions of minimal supergravity in five dimensions. Moreover, such solutions were further lifted to M-theory in a simple manner. Subsequent work [4] made the remarkable observation that metrics of the Gödel type are T-dual to plane waves. It was demonstrated there that a wide class of type-IIA Gödel-like solutions could be found by considering the T-duals of various type-IIB plane wave solutions. This was later applied in [5] to produce an array of supersymmetric Gödel universes in string theory.

One would expect, therefore, that any problems arising from CTCs on the Gödel side of the T-duality would have a mirror on the plane wave side. It is important to note, however, that problems on the plane wave side appear only after the compactification involved in the T-duality operation. After all, plane waves are well-defined string backgrounds with no apparent conceptual problems. CTCs appear in *compact* plane waves, however [6], and in quite some generality [7]. Whilst one might hope that string theory somehow resolves any

JHEP11(2003)033

conceptual difficulties related to CTCs, there is evidence to suggest that string propagation on such compact plane waves — and so also on Gödel backgrounds — is problematic [8], although the issues are somewhat subtle.

Another example of metrics with CTCs which has been analysed within the framework of string theory, is the so called BMPV black hole [9] (see also [10]). Here, however, the issues are well understood. There is a bound on the angular momentum of the black hole which, if satisfied, gives a solution with CTCs, but which are entirely hidden behind the horizon. Only in the over-rotating case do the CTCs appear outside of the would-be horizon, although the solution in that case does not describe a black hole as such [11]. These latter backgrounds, which potentially have CTCs throughout the spacetime, have indeed been shown to be unphysical for various reasons [11]–[14], and can therefore be safely discarded. Whether the CTCs in Gödel-like backgrounds can be resolved in such ways is, however, an important unanswered question. In [4] it was suggested that holography might play an important role, in particular that holographic screens might shield the CTCs, although the validity of this proposal is under debate [15, 6]. Other attempts to exclude Gödel-like backgrounds on physical grounds include [16]. In the sense that the CTCs associated with the Gödel background could possibly be removed or avoided they are not as pathological as those associated with the over-rotating black hole.

The aim of this paper is to generalise the (three-charge) BMPV black hole to a spacetime which is asymptotic to the five-dimensional Gödel universe, yet retains the horizon. In agreement with earlier results [12, 17, 18] we show that the Gödel deformation of the black hole becomes irrelevant close to the horizon. The horizon area and entropy is unchanged from the asymptotically flat case. Whereas such a solution has been found already in [18], we work within the context of dimensional reduction from ten dimensions, so obtain from the outset an explicit ten-dimensional microscopic description of this "BMPV Gödel black hole". Such a description would be hard to find from the solution of [18] directly and, indeed, we show that our microscopic understanding of this five-dimensional solution has various important implications. Moreover, although we will not analyse it in much detail, we actually find a more general BMPV Gödel black hole to that considered in [18] and, again, it should be emphasised that this comes about only through the ten-dimensional description. We should also note that all the solutions we consider will be extremal. We leave the non-extremal generalisation of our solution for future work; it would be interesting to see if applying our method to this case gives the non-extremal solution of [18].

Our asymptotically Gödel black hole allows, amongst other things, for a more detailed investigation of holographic screens and CTCs. In particular, it is shown that as long as one stays below an upper bound on the angular momentum for the black hole, there will always be a causally safe region inside the holographic screen. Moreover, only when one violates this bound on the angular momentum, thereby creating CTCs just outside the black hole horizon, can such backgrounds exhibit causally sick behaviour *everywhere*.

The CTCs associated with the black hole, and those associated with the Gödel background, are of a very different nature. We hope that our solution, which combines both types of CTC, will be useful in shedding light on how, if at all, the CTCs in the Gödel universe can be resolved.

JHEP11(2003)033

In the following section, we generate a solution of type-IIB supergravity which describes rotating D1-D5-branes in a plane wave background.[1] As in the non-rotating case considered in [22], this is achieved by applying the solution-generating method [23, 24, 25] of Garfinkle and Vachaspati (GV) to the standard solution describing rotating D1-D5-branes. This technique has been known for some time, though has most often been used to generate asymptotically pp-wave solutions. To the best of our knowledge, the first time it was used to generate asymptotically plane wave solutions was in [26]. In section three, we T-dualise this to generate a type-IIA solution describing rotating branes in a Gödel background, many of the properties of which are inherited by our five-dimensional Gödel black hole. We construct this in section four by dimensional reduction of the ten-dimensional solution. In section five, we analyse the properties of this black hole, discussing its horizon area, the associated bound on the angular momentum of the hole, the existence of CTCs and the possible holographic protection of chronology. We conclude in section six.

We include two appendices describing our conventions. The first lists the field equations of the ten-dimensional type-IIB and IIA supergravity theories, as well as the relevant T-duality rules which map between them. The second describes the dimensional reduction of the IIB theory to five dimensions, generalising somewhat the analysis of [27].

2. Asymptotically plane wave rotating branes

We begin with a type-IIB solution representing intersecting D1-D5-branes with non-trivial angular momentum[2] [12]:

$$\begin{aligned} \mathrm{d}s^2 &= H_1^{-\frac{3}{4}} H_5^{-\frac{1}{4}} \left[-\mathrm{d}t^2 + \mathrm{d}y_5^2 + \frac{J}{r^2}\left(\cos^2\theta\,\mathrm{d}\phi_1 - \eta\sin^2\theta\,\mathrm{d}\phi_2\right)(\mathrm{d}t - \mathrm{d}y_5)\right] + \\ &\quad + H_1^{\frac{1}{4}} H_5^{\frac{3}{4}} \mathrm{d}s^2(\mathbb{R}^4) + H_1^{\frac{1}{4}} H_5^{-\frac{1}{4}} \mathrm{d}s^2(T^4)\,, && (2.1)\\ C_2 &= \left(H_1^{-1} - 1\right)\mathrm{d}t\wedge\mathrm{d}y_5 - \frac{J}{2r^2} H_1^{-1}(\mathrm{d}t - \mathrm{d}y_5)\wedge\left(\cos^2\theta\,\mathrm{d}\phi_1 - \eta\sin^2\theta\,\mathrm{d}\phi_2\right) - \\ &\quad - \eta Q_5 \cos^2\theta\,\mathrm{d}\phi_1\wedge\mathrm{d}\phi_2\,, && (2.2)\\ e^{2\Phi} &= H_1 H_5^{-1}\,, && (2.3) \end{aligned}$$

where, in terms of the coordinates $y_1, \ldots, y_4$, $\mathrm{d}s^2(T^4)$ is the flat metric on T^4, the metric on $\mathbb{R}^4$ is

$$\mathrm{d}s^2(\mathbb{R}^4) = \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \cos^2\theta\,\mathrm{d}\phi_1^2 + \sin^2\theta\,\mathrm{d}\phi_2^2\right) \equiv \sum_{i=1}^{4} \mathrm{d}z_i^2\,, \qquad (2.4)$$

and the functions associated with the D1- and D5-branes are

$$H_1 = 1 + \frac{Q_1}{r^2}\,, \qquad H_5 = 1 + \frac{Q_5}{r^2}\,. \qquad (2.5)$$

[1] Other solutions describing (intersecting) D-branes in plane wave backgrounds have been considered in [19, 20, 21].

[2] The Brinkmann, or pp-wave, term discussed in [12] is not included here as it will be generated alongside the plane wave term in the subsequent analysis. Relative to the solution found in [12], we have absorbed a sign into C_2 and switched $\phi_1 \leftrightarrow \phi_2$, so that the polar parametrisation of $\mathbb{R}^4$ is given by $z_1 + iz_2 = r\cos\theta\, e^{i\phi_1}$, $z_3 + iz_4 = r\sin\theta\, e^{i\phi_2}$.

JHEP11(2003)033

We have further introduced an arbitrary parameter, $\eta = \pm 1$, into the solution. This allows one to consider either D5-branes ($\eta = +1$ and positive charge), or anti-D5-branes ($\eta = -1$ and negative charge). As noted in [18], the $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry of the metric (2.1), allows one to define a left, J_L, and a right, J_R, angular momentum. They are just the Casimir operators of $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$. Then, since the rotation one-form,

$$\cos^2\theta\,\mathrm{d}\phi_1 - \eta\sin^2\theta\,\mathrm{d}\phi_2\,, \tag{2.6}$$

appearing in the metric (2.1) is a right (left) one-form of SU(2) for $\eta = +1$ ($\eta = -1$), the solution with $\eta = +1$ has non-zero J_R, whereas that with $\eta = -1$ has non-zero J_L.

To generate the asymptotically plane wave generalisation of this seed solution, we work with lightcone coordinates

$$u = \frac{1}{2}(t - y_5)\,, \qquad v = \frac{1}{2}(t + y_5)\,, \tag{2.7}$$

so that $\partial/\partial u$ is a null Killing vector. We can then exploit a fact demonstrated by GV: the presence of such a Killing vector leads to the separation of the Ricci tensor into a standard and a plane wave component [23, 24, 25]. The appropriate method is discussed by Liu *et al* [22] who, amongst other things, considered the non-rotating D1-D5-brane solution, showing that the $r = 0$ horizon of the seed solution is preserved by the GV procedure. However, it is not clear from the analysis of [22] that the horizon is *regular*.[3] Indeed we would expect, as in the simpler examples considered in [28], that the deformation we introduce actually destabilises the would-be horizon, giving rise to a pp-curvature singularity at $r = 0$. In the non-rotating case, this has been confirmed in [21], and it seems unlikely that the addition of rotation changes this behaviour. Of course, the process of T-duality and/or dimensional reduction to give an asymptotically Gödel solution, would "remove" such singular behaviour, giving rise to a perfectly regular horizon at $r = 0$. Note that there are certain types of plane wave deformations which do *not* give rise to such singularities [26], but these are somewhat more complicated than those considered here. At any rate, relative to the analysis of [22], we will find that we can add rotation for free.

As in [22], we write

$$\begin{aligned}\mathrm{d}s^2 \;=&\; H_1^{-\frac{3}{4}} H_5^{-\frac{1}{4}} \left[-4\mathrm{d}u\,\mathrm{d}v + \mathcal{H}\mathrm{d}u^2 + \frac{2J}{r^2}\left(\cos^2\theta\,\mathrm{d}\phi_1 - \eta\sin^2\theta\,\mathrm{d}\phi_2\right)\mathrm{d}u\right] + \\ &+ H_1^{\frac{1}{4}} H_5^{\frac{3}{4}} \mathrm{d}s^2(\mathbb{R}^4) + H_1^{\frac{1}{4}} H_5^{-\frac{1}{4}} \mathrm{d}s^2(T^4)\,.\end{aligned} \tag{2.8}$$

where, asymptotically, we want $\mathcal{H}$ to be a quadratic function of the transverse coordinates,

$$\mathcal{H} = \sum_{i,j=1}^{4} \left(A_{ij} y_i y_j + B_{ij} z_i z_j\right). \tag{2.9}$$

The behaviour of $\mathcal{H}$ will be further restricted by the analysis which follows. The relevant equations of motion are as in (A.2). We must support the plane wave term with an

[3]We thank Mukund Rangamani and Nobuyoshi Ohta for a discussion of this point.

appropriate contribution to the stress-energy tensor so that the Einstein equations for our solution are still satisfied. Since the Ricci tensor separates, the plane wave term affects only the uu-component of the modified Ricci tensor, $\mathcal{R}_{ab}$. The natural candidate to support this modification of the Einstein equations is a null five-form.[4]

The five-form must also be self-dual and satisfy $F_5 \wedge F_3 = 0$. Two natural choices are [22, 5]

$$F_5 = \mu\, \mathrm{d}u \wedge (\mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \mathrm{d}z_1 \wedge \mathrm{d}z_2 + \mathrm{d}y_3 \wedge \mathrm{d}y_4 \wedge \mathrm{d}z_3 \wedge \mathrm{d}z_4)\,, \tag{2.10}$$

$$F_5 = \frac{\mu}{\sqrt{2}}\mathrm{d}u \wedge (\mathrm{d}y_1 \wedge \mathrm{d}y_2 + \mathrm{d}y_3 \wedge \mathrm{d}y_4) \wedge (\mathrm{d}z_1 \wedge \mathrm{d}z_2 + \mathrm{d}z_3 \wedge \mathrm{d}z_4)\,. \tag{2.11}$$

The dilaton and F_3 equations are unchanged, and all that remains to be checked is the Einstein equation. The Ricci tensor of our asymptotically plane wave metric is given in terms of the unperturbed Ricci tensor, $\overline{\mathcal{R}}_{ab}$, by

$$\mathcal{R}_{ab} = \overline{\mathcal{R}}_{ab} - \delta^u_a \delta^u_b \frac{1}{2H_1 H_5}\left(\Box + H_5 \widehat{\Box} + \frac{1}{4}(h_5^2 + 3\,h_1^2)\right)\mathcal{H}\,, \tag{2.12}$$

where

$$h_1 = \frac{1}{H_1}\frac{\partial}{\partial r}H_1\,, \qquad h_5 = \frac{1}{H_5}\frac{\partial}{\partial r}H_5\,, \tag{2.13}$$

and $\Box$, $\widehat{\Box}$ are the laplacian operators on $\mathbb{R}^4$, T^4 respectively. The normalisation of F_5 in (2.10) and (2.11) has been chosen in such a way that, for both,

$$F_{ac_1\ldots c_4}F_b{}^{c_1\ldots c_4} = \frac{48\mu^2}{H_1 H_5}\delta^u_a \delta^u_b\,. \tag{2.14}$$

The $\mathcal{H}$-deformed components of the terms arising from F_3 are

$$F_{uab}F_u{}^{ab} = \frac{16J^2}{r^8 H_1^{\frac{5}{2}} H_5^{\frac{3}{2}}} - \frac{2h_1^2}{H_1^{\frac{3}{2}} H_5^{\frac{1}{2}}}\mathcal{H}\,, \qquad g_{uu}F_3^2 = \frac{6\left(h_5^2 - h_1^2\right)}{H_1^{\frac{3}{2}} H_5^{\frac{1}{2}}}\mathcal{H}\,, \tag{2.15}$$

and one can thus show that the uu-component of the Einstein equations is satisfied provided

$$\left(\Box + H_5 \widehat{\Box}\right)\mathcal{H} = -\mu^2\,. \tag{2.16}$$

Despite the addition of angular momentum, this equation is unchanged from that derived by Liu *et al* [22] in the non-rotating case.

The general spherically symmetric solution to this Poisson equation is a function of the radii, r and $\hat{r}$, of $\mathbb{R}^4$ and T^4 respectively. One can thus find a general solution which asymptotes to the maximally supersymmetric BFHP plane wave [29],

$$\mathcal{H} = \frac{4Q_k}{r^2} - \frac{\mu^2}{16}\left(r^2 + \hat{r}^2 - 4Q_5 \ln r\right)\,, \tag{2.17}$$

just as in [22]. Our objective is to compactify on the four-torus, however, so we only

[4] There are obviously other possibilities, though we will not consider them here.

consider a solution $\mathcal{H} = \mathcal{H}(r)$, taking

$$\mathcal{H} = \frac{4Q_k}{r^2} - \frac{\mu^2}{8} r^2 \,. \tag{2.18}$$

One could clearly also include an additive constant in the solution for $\mathcal{H}$, but this may subsequently be absorbed by shifts in u.

Our asymptotically plane wave solution is thus

$$\begin{aligned} \mathrm{d}s^2 = & \; H_1^{-\frac{3}{4}} H_5^{-\frac{1}{4}} \left[-4\mathrm{d}u\,\mathrm{d}v + \left(\frac{4Q_k}{r^2} - \frac{\mu^2}{8} r^2 \right) \mathrm{d}u^2 + \frac{2J}{r^2} \left(\cos^2\theta\,\mathrm{d}\phi_1 - \eta \sin^2\theta\,\mathrm{d}\phi_2 \right) \mathrm{d}u \right] + \\ & + H_1^{\frac{1}{4}} H_5^{\frac{3}{4}} \mathrm{d}s^2(\mathbb{R}^4) + H_1^{\frac{1}{4}} H_5^{-\frac{1}{4}} \mathrm{d}s^2(T^4) \,, \end{aligned} \tag{2.19}$$

supported by the dilaton, (2.3), the RR two-form potential, (2.2), and an RR four-form potential C_4 giving rise to one of the five-forms, $F_5 = \mathrm{d}C_4$, in (2.10) or (2.11). Switching off the five-form field strength by setting $\mu = 0$, we obtain the rotating D1-D5-pp-wave system given explicitly in [12]. This is the ten-dimensional description of the familiar five-dimensional rotating three-charge black hole [30, 31], which is obtained upon dimensional reduction along the five directions $y_1, \ldots y_5$. Setting all three charges to be equal, $Q_1 = Q_5 = Q_k$, we recover the BMPV black hole [9].

If instead we switch off the black hole charges, setting $Q_1 = Q_5 = Q_k = J = 0$, then we obtain a standard supersymmetric plane wave as expected. The observation [4, 5] that this is T-dual to a type IIA Gödel universe, $\mathcal{G}_5 \times \mathbb{R}^5$, implies that the solution presented here will be T-dual to a rotating D0-D4-F1 system in a Gödel universe. We will demonstrate that this is indeed the case in the next section.

3. Rotating branes in a rotating universe

3.1 Constructing the IIA solution

In order to T-dualise this solution, we must choose an everywhere spacelike Killing vector, K, along which to perform the T-duality. To generate a Gödel-like universe one takes K to be a translation in y_5 plus rotations in the transverse planes [4, 5]. For the plane wave, the quotient by the action of such a K leads to CTCs [6] which are inherited by the Gödel solution. More generally, the quotient of a plane wave by the action of any Killing vector with a $\partial/\partial u$ component will give rise to a Gödel-like universe with CTCs [7], and we expect the same to be true here. We thus take

$$K = \frac{\partial}{\partial y_5} - \left(\beta_1 \frac{\partial}{\partial \phi_1} + \beta_2 \frac{\partial}{\partial \phi_2} \right), \tag{3.1}$$

which has norm

$$|K|^2 = 1 + \zeta(r,\theta) \equiv H_{k\beta}(r,\theta), \tag{3.2}$$

where

$$\zeta(r,\theta) = \left(H_1 H_5 (\beta_1^2 \cos^2\theta + \beta_2^2 \sin^2\theta) - \frac{\mu^2}{32} \right) r^2 + \frac{Q_k}{r^2} + \frac{J}{r^2} \left(\beta_1 \cos^2\theta - \eta \beta_2 \sin^2\theta \right). \tag{3.3}$$

The simplest way to ensure $|K|^2 > 0$ everywhere is to take $\beta_1^2 = \beta_2^2 = \mu^2/32 = \beta^2$. Defining $\beta_1 = \beta$, $\beta_2 = \epsilon\beta$, where $\epsilon = \pm 1$, we have

$$\zeta(r,\theta) = \beta^2(Q_1 + Q_5) + \frac{Q_{k\beta}(\theta)}{r^2}\,, \tag{3.4}$$

where

$$Q_{k\beta}(\theta) = Q_k + \beta^2 Q_1 Q_5 + \beta J\left(\cos^2\theta - \epsilon\eta\sin^2\theta\right). \tag{3.5}$$

With both β and J positive, and with the choice $\epsilon = -\eta$, no further restriction is required. In this case, the twist induced by K has the same sign relative to the original angular momentum, governed by J, in each of the two planes of rotation. With $\epsilon = +\eta$, however, this is not the case, and we need to impose a further restriction,

$$Q_k + \beta^2 Q_1 Q_5 \geq \beta J\,, \tag{3.6}$$

on the charges to ensure that K is everywhere spacelike. (Note that if we instead take either β or J to be negative, then the same condition should also be imposed.) This choice further introduces a θ dependence in the metric. Either way, we can now T-dualise along the orbits of K. For simplicity, when studying solutions which contain both background Gödel rotation, β, and rotation, J, of the branes, we will usually restrict to the case $\epsilon = -\eta$ (although see the discussion of minimal supergravity in the following section). This corresponds to taking the two types of rotations, induced by β and J, to contribute to the same angular momentum, J_R (J_L) for $\eta = +1$ (-1) respectively.

Introducing a new pair of angular coordinates

$$\widetilde{\phi}_1 = \phi_1 - 2\beta_1 u\,, \qquad \widetilde{\phi}_2 = \phi_2 - 2\beta_2 u\,, \tag{3.7}$$

which satisfy $K(\widetilde{\phi}_i) = 0$, the Killing vector along which we T-dualise becomes $\partial/\partial y_5$. With $\mathrm{d}s^2(\widetilde{\mathbb{R}}^4)$ denoting the metric on $\mathbb{R}^4$ in terms of these new coordinates, the resulting metric is

$$\begin{aligned}\mathrm{d}s^2 \,=\,& H_1^{-\frac{3}{4}} H_5^{-\frac{1}{4}} H_{k\beta}\left(\mathrm{d}y_5 - H_{k\beta}^{-1}(\zeta(r,\theta)\mathrm{d}t + \sigma)\right)^2 + H_1^{\frac{1}{4}} H_5^{\frac{3}{4}} \mathrm{d}s^2(\widetilde{\mathbb{R}}^4) - \\ & - H_1^{-\frac{3}{4}} H_5^{-\frac{1}{4}} H_{k\beta}^{-1}\left(\mathrm{d}t - \sigma\right)^2 + H_1^{\frac{1}{4}} H_5^{-\frac{1}{4}} \mathrm{d}s^2(T^4)\,,\end{aligned} \tag{3.8}$$

where we have returned to the coordinates (t, y_5). The metric has been written in a way adapted to dimensional reduction along y_5, and we will utilise this in the following section. Here we T-dualise according to the equations (A.3). We have also defined the one-form

$$\sigma = r^2\left(\gamma_1(r)\cos^2\theta\,\mathrm{d}\widetilde{\phi}_1 + \gamma_2(r)\sin^2\theta\,\mathrm{d}\widetilde{\phi}_2\right), \tag{3.9}$$

in which

$$\gamma_1(r) = \beta_1 H_1 H_5 + \frac{J}{2r^4}\,, \qquad \gamma_2(r) = \beta_2 H_1 H_5 - \frac{\eta J}{2r^4}\,. \tag{3.10}$$

The RR two-form potential becomes

$$C_2 = \left(\left(H_1^{-1} - 1\right)\mathrm{d}t + \omega\right)\wedge \mathrm{d}y_5 + \mathrm{d}t\wedge\omega - \eta\, Q_5\cos^2\theta\,\mathrm{d}\widetilde{\phi}_1\wedge\mathrm{d}\widetilde{\phi}_2\,, \tag{3.11}$$

where we define the one-form

$$\omega = f_2(r,\theta)\mathrm{d}\widetilde{\phi}_1 - \eta f_1(r,\theta)\mathrm{d}\widetilde{\phi}_2\,, \tag{3.12}$$

in which

$$f_1(r,\theta) = \beta_1 Q_5\cos^2\theta - \frac{J}{2r^2}H_1^{-1}\sin^2\theta\,, \qquad f_2(r,\theta) = \left(\eta\beta_2 Q_5 - \frac{J}{2r^2}H_1^{-1}\right)\cos^2\theta\,. \tag{3.13}$$

With the five-form field strength given by either (2.10) or (2.11), there is a gauge freedom in choosing a four-form such that $F_5 = \mathrm{d}C_4$. Concentrating on the five-form (2.10) for definiteness, in many ways the simplest choice is to take

$$C_4 = \frac{\mu(t-y_5)}{2}\left(\mathrm{d}y_1\wedge\mathrm{d}y_2\wedge\mathrm{d}\tilde{z}_1\wedge\mathrm{d}\tilde{z}_2 + \mathrm{d}y_3\wedge\mathrm{d}y_4\wedge\mathrm{d}\tilde{z}_3\wedge\mathrm{d}\tilde{z}_4\right)\,, \tag{3.14}$$

which does not contribute to the three-form RR potential of the IIA solution, giving a five-form alone. On the other hand, we could choose

$$C_4 = -\frac{\mu}{2}\left(\mathrm{d}t - \mathrm{d}y_5\right)\wedge\left(y_1\,\mathrm{d}y_2\wedge\mathrm{d}\tilde{z}_1\wedge\mathrm{d}\tilde{z}_2 + y_3\,\mathrm{d}y_4\wedge\mathrm{d}\tilde{z}_3\wedge\mathrm{d}\tilde{z}_4\right)\,, \tag{3.15}$$

which gives rise to both three-form and five-form RR potentials in the IIA solution.

With C_4 as in (3.14), the resulting IIA solution is

$$\begin{aligned}
\mathrm{d}s^2 &= -H_1^{-\frac{7}{8}}H_5^{-\frac{3}{8}}H_{k\beta}^{-\frac{3}{4}}\left(\mathrm{d}t-\sigma\right)^2 + H_1^{\frac{1}{8}}H_5^{\frac{5}{8}}H_{k\beta}^{\frac{1}{4}}\mathrm{d}s^2(\widetilde{\mathbb{R}}^4) + H_1^{\frac{1}{8}}H_5^{-\frac{3}{8}}H_{k\beta}^{\frac{1}{4}}\mathrm{d}s^2(T^4) + \\
&\quad + H_1^{\frac{1}{8}}H_5^{\frac{5}{8}}H_{k\beta}^{-\frac{3}{4}}\mathrm{d}y_5^2\,,\\
e^{2\Phi} &= H_1^{\frac{3}{2}}H_5^{-\frac{1}{2}}H_{k\beta}^{-1}\,,\\
B_2 &= \left(\left(H_{k\beta}^{-1}-1\right)\mathrm{d}t - H_{k\beta}^{-1}\sigma\right)\wedge\mathrm{d}y_5\,, \qquad C_1 = -\left(H_1^{-1}-1\right)\mathrm{d}t - \omega\,,\\
C_3 &= \left(r^2 H_{k\beta}^{-1}\left(\eta\gamma_1 f_1\cos^2\theta + \gamma_2 f_2\sin^2\theta\right) - \eta Q_5\cos^2\theta\right)\mathrm{d}\widetilde{\phi}_1\wedge\mathrm{d}\widetilde{\phi}_2\wedge\mathrm{d}y_5 + \\
&\quad + H_{k\beta}^{-1}\mathrm{d}t\wedge\left(\omega - Q_1 H_1^{-1}r^{-2}\sigma\right)\wedge\mathrm{d}y_5\,,\\
C_5 &= \frac{\mu(t-y_5)}{2}\left(\mathrm{d}y_1\wedge\mathrm{d}y_2\wedge\mathrm{d}\tilde{z}_1\wedge\mathrm{d}\tilde{z}_2 + \mathrm{d}y_3\wedge\mathrm{d}y_4\wedge\mathrm{d}\tilde{z}_3\wedge\mathrm{d}\tilde{z}_4\right)\wedge\mathrm{d}y_5\,.
\end{aligned} \tag{3.16}$$

The NS-NS two form, B_2, gives rise to a field strength, $H_3 = \mathrm{d}B_2$, and the RR field strengths are $F_2 = \mathrm{d}C_1$ and $G_4 = \mathrm{d}C_3 + C_1\wedge H_3$. There is also a six-form field strength, $G_6 = \mathrm{d}C_5 + C_3\wedge H_3$, which, as described in appendix A, can be dualised as $G_4' = -e^{-\Phi/2}\star G_6$ to give a further contribution to the four-form field strength. The other choice of five-form, as in (2.11), and with a similar choice of gauge as above, gives rise to an alternative IIA solution with

$$C_5 = \frac{\mu(t-y_5)}{2\sqrt{2}}\left(\mathrm{d}y_1\wedge\mathrm{d}y_2 + \mathrm{d}y_3\wedge\mathrm{d}y_4\right)\wedge\left(\mathrm{d}\tilde{z}_1\wedge\mathrm{d}\tilde{z}_2 + \mathrm{d}\tilde{z}_3\wedge\mathrm{d}\tilde{z}_4\right)\wedge\mathrm{d}y_5\,. \tag{3.17}$$

JHEP11(2003)033

3.2 Dissecting the IIA solution

As is easily seen by considering the limits in which more familiar supergravity solutions arise, the type-IIA solution (3.16) corresponds to a rotating D0-D4-F1 system in a Gödel universe. It has CTCs for radii

$$r^2\gamma_i^2 \geq H_1 H_5 H_{k\beta}\,, \quad i = 1,2\,. \tag{3.18}$$

Compactification on $T^4 \times S^1$ will not change this property of the solution, so we expect it to persist in the five-dimensional asymptotically Gödel black hole solutions of the following section. We do indeed find the same condition there, so will postpone its analysis until then.

Switching off the Gödel parameter, taking $\beta = 0$, gives a solution discussed in [27, 14]:

$$\begin{aligned}
\mathrm{d}s^2 &= -H_1^{-\frac{7}{8}} H_5^{-\frac{3}{8}} H_k^{-\frac{3}{4}} (\mathrm{d}t - \sigma)^2 + H_1^{\frac{1}{8}} H_5^{\frac{5}{8}} H_k^{\frac{1}{4}} \mathrm{d}s^2(\widetilde{\mathbb{R}}^4) + H_1^{\frac{1}{8}} H_5^{-\frac{3}{8}} H_k^{\frac{1}{4}} \mathrm{d}s^2(T^4) + \\
&\quad + H_1^{\frac{1}{8}} H_5^{\frac{5}{8}} H_k^{-\frac{3}{4}} \mathrm{d}y_5^2\,, \\
e^{2\Phi} &= H_1^{\frac{3}{2}} H_5^{-\frac{1}{2}} H_k^{-1}\,, \\
B_2 &= \left(\left(H_k^{-1} - 1\right) \mathrm{d}t - H_k^{-1}\sigma\right) \wedge \mathrm{d}y_5\,, \\
C_1 &= -\left(H_1^{-1} - 1\right) \mathrm{d}t + H_1^{-1}\sigma\,, \\
C_3 &= H_k^{-1} \mathrm{d}t \wedge \mathrm{d}y_5 \wedge \sigma\,,
\end{aligned} \tag{3.19}$$

where

$$\sigma = \frac{J}{2r^2}\left(\cos^2\theta \mathrm{d}\tilde{\phi}_1 - \eta \sin^2\theta \mathrm{d}\tilde{\phi}_2\right), \tag{3.20}$$

and

$$H_k = 1 + \frac{Q_k}{r^2}\,. \tag{3.21}$$

It describes a rotating D0-D4-F1 system, with (D0,D4,F1) charges (Q_1, Q_5, Q_k). Compactification on the T^4 would give rise to the five-dimensional rotating three-charge black hole and, as in that case [27, 14], the condition for the existence of CTCs reduces to

$$J^2 \geq 4r^6 H_1 H_5 H_k\,. \tag{3.22}$$

The other limiting case is pure Gödel: $Q_1 = Q_5 = Q_k = J = 0$. In terms of the field strengths $H_3 = \mathrm{d}B_2$ and $F_4 = \star F_6 = \star(\mathrm{d}C_5)$, the solution is

$$\begin{aligned}
\mathrm{d}s^2 &= -\left(\mathrm{d}t - \sigma\right)^2 + \mathrm{d}s^2(\tilde{\mathbb{R}}^4) + \mathrm{d}s^2(T^4) + \mathrm{d}y_5^2\,, \\
H_3 &= -2\beta\left(\mathrm{d}\tilde{z}^1 \wedge \mathrm{d}\tilde{z}^2 + \epsilon \mathrm{d}\tilde{z}^3 \wedge \mathrm{d}\tilde{z}^4\right) \wedge \mathrm{d}y_5\,, \\
F_4 &= 2\sqrt{2}\beta\left(\mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \mathrm{d}\tilde{z}_1 \wedge \mathrm{d}\tilde{z}_2 + \mathrm{d}y_3 \wedge \mathrm{d}y_4 \wedge \mathrm{d}\tilde{z}_3 \wedge \mathrm{d}\tilde{z}_4\right),
\end{aligned} \tag{3.23}$$

where now

$$\sigma = \beta r^2(\cos^2\theta \mathrm{d}\tilde{\phi}_1 + \epsilon \sin^2\theta \mathrm{d}\tilde{\phi}_2)\,. \tag{3.24}$$

The parity of the rotation in the original Killing vector (3.1) thus determines whether we generate a Gödel universe with non-zero J_L or J_R. In either case the condition for the existence of CTCs reduces to

$$r^2 > \frac{1}{\beta^2}\,. \tag{3.25}$$

We will refer to the surface at which such CTCs (associated with the Gödel rotation) appear as the velocity of light (VL) surface.

The general case has non-zero Gödel rotation, β, and "bare" brane rotation, J. (Recall that we will set $\epsilon = -\eta$, so that both rotations contribute to the single non-zero angular momentum, J_L or J_R.) The metric has the same form as that of a D0-D4-F1 system, but with a modified one-form, σ, governing rotation and a modified harmonic function, $H_{k\beta}$, associated with the fundamental string. The one-form is

$$\sigma = \left(\beta r^2 + \beta(Q_1+Q_5) + \frac{J+2\beta Q_1 Q_5}{2r^2}\right)(\cos^2\theta\,\mathrm{d}\tilde{\phi}_1 - \eta\sin^2\theta\,\mathrm{d}\tilde{\phi}_2)\,, \tag{3.26}$$

in which the $\mathcal{O}(r^2)$ term is the Gödel rotation, and the $\mathcal{O}(1/r^2)$ term is the rotation of the branes. It is thus clear that frame dragging effects associated with the background Gödel rotation modifies the bare brane rotation to give an effective rotation,

$$J_\beta = J + 2\beta Q_1 Q_5\,. \tag{3.27}$$

Consider now the harmonic function associated with the fundamental string,

$$H_{k\beta}(r) = \lambda\left(1 + \frac{\lambda^{-1}Q_{k\beta}}{r^2}\right), \tag{3.28}$$

where

$$\lambda = 1 + \beta^2(Q_1+Q_5) \qquad \text{and} \qquad Q_{k\beta} = Q_k + \beta^2 Q_1 Q_5 + \beta J\,. \tag{3.29}$$

The Gödel parameter combines with the bare rotation, and D0- and D4-brane charges, Q_1 and Q_5, to give a modified string charge $\lambda^{-1}Q_{k\beta}$.

At this stage the richness of the ten-dimensional solution is apparent. Compactification on $T^4 \times S^1$ gives a five-dimensional solution with equally rich properties.

4. The five-dimensional rotating Gödel black hole

By choosing the initial plane wave deformation, $\mathcal{H}$, to be independent of the transverse coordinates, $y_1, \ldots, y_4$, we have preserved translations in these directions as isometries of our final solution. We can thus compactify either the IIB or the IIA solution on the $T^4 \times S^1$ provided by the directions $y_1 \ldots y_5$. Although this will give rise to the same five-dimensional solution, we follow the type-IIB route described in [27]. Of course in that case, we perform a twisted compactification along y_5, so will compactify the rotated solution (3.8), (3.11) and (2.3) on $T^4 \times S^1$ in the usual way.

The compactification ansatz derived in [27] is reviewed in appendix B. In addition to the dilaton, which reduces trivially, this generates a further two scalars, ψ and χ, a gauge field, A_1, coming from the metric, and two gauge fields, B_2 and C_1, coming from the RR two-form. We find the five-dimensional solution

$$
\begin{aligned}
\mathrm{d}s_5^2 &= -\left(H_1 H_5 H_{k\beta}\right)^{-\frac{2}{3}} (\mathrm{d}t - \sigma)^2 + \left(H_1 H_5 H_{k\beta}\right)^{\frac{1}{3}} \mathrm{d}s^2(\widetilde{\mathbb{R}}^4)\,, \\
A_1 &= -H_{k\beta}^{-1} \left(\zeta \mathrm{d}t + \sigma\right), \\
B_2 &= -\eta Q_5 \cos^2\theta \mathrm{d}\tilde{\phi}_1 \wedge \mathrm{d}\tilde{\phi} + \mathrm{d}t \wedge \omega, \\
C_1 &= (H_1^{-1} - 1)\mathrm{d}t + \omega\,,
\end{aligned}
\tag{4.1}
$$

$$
e^{2\Phi} = H_1 H_5^{-1}\,, \qquad e^{2b\psi} = H_1^{-\frac{1}{3}} H_5^{\frac{1}{3}}\,, \qquad e^{2a\chi} = (H_1 H_5)^{-\frac{1}{6}} H_{k\beta}^{\frac{1}{3}}, \tag{4.2}
$$

where $a^2 = 1/24$, $b^2 = 1/9$ and the one-form, σ, is given by (3.9). Note that we have $\Phi = -3b\psi$, which is consistent with the equations of motion (B.11); there are effectively only two scalar fields. The gauge potentials combine to give the field strengths $F_{(1)} = \mathrm{d}A_1$, $F_{(2)} = \mathrm{d}C_1$ and $G_3 = \mathrm{d}B_2 - F_{(2)} \wedge A_1$. The latter can be dualised as

$$
G_3 = e^{-(\Phi + 2(2a\chi - 3b\psi/2))} \star \mathrm{d}\hat{A}_1 \equiv e^{-(\Phi + 2(2a\chi - 3b\psi/2))} \star \hat{F}_2\,. \tag{4.3}
$$

In addition, there are three-form and two-form field strengths coming from the reduction of the RR five-form. As described in appendix B, the choice (2.10) gives rise to

$$
F_3^{ij} = \frac{\mu}{2} \mathrm{d}t \wedge \mathrm{d}\tilde{z}_i \wedge \mathrm{d}\tilde{z}_j\,, \qquad F_2^{ij} = -\frac{\mu}{2} \mathrm{d}\tilde{z}_i \wedge \mathrm{d}\tilde{z}_j\,, \tag{4.4}
$$

for $i, j = 1, 2$ and $i, j = 3, 4$ only. They combine to give the gauge invariant field strength

$$
G_3^{ij} = F_3^{ij} - F_2^{ij} \wedge A_1 = \frac{\mu}{2} H_{k\beta}^{-1} (\mathrm{d}t - \sigma) \wedge \mathrm{d}\tilde{z}_i \wedge \mathrm{d}\tilde{z}_j\,, \tag{4.5}
$$

which can be dualised by taking

$$
G_3^{ij} = e^{-4a\chi} \star \mathrm{d}\hat{A}_1^{ij}\,, \tag{4.6}
$$

in terms of the dual potentials

$$
\hat{A}_1^{12} = \frac{\mu}{4} \left(\tilde{z}_3 \mathrm{d}\tilde{z}_4 - \tilde{z}_4 \mathrm{d}\tilde{z}_3\right), \qquad \hat{A}_1^{34} = \frac{\mu}{4} \left(\tilde{z}_1 \mathrm{d}\tilde{z}_2 - \tilde{z}_2 \mathrm{d}\tilde{z}_1\right). \tag{4.7}
$$

On the other hand, the choice (2.11) gives

$$
F_3^{12} = F_3^{34} = \frac{\mu}{2\sqrt{2}} \mathrm{d}t \wedge (\mathrm{d}\tilde{z}_1 \wedge \mathrm{d}\tilde{z}_2 + \mathrm{d}\tilde{z}_3 \wedge \mathrm{d}\tilde{z}_4)\,, \qquad F_2^{12} = F_2^{34} = -\frac{\mu}{2\sqrt{2}} (\mathrm{d}\tilde{z}_1 \wedge \mathrm{d}\tilde{z}_2 + \mathrm{d}\tilde{z}_3 \wedge \mathrm{d}\tilde{z}_4)\,. \tag{4.8}
$$

They combine to give the gauge invariant field strength

$$
G_3^{12} = G_3^{34} = \frac{\mu}{2\sqrt{2}} H_{k\beta}^{-1} (\mathrm{d}t - \sigma) \wedge (\mathrm{d}\tilde{z}_1 \wedge \mathrm{d}\tilde{z}_2 + \mathrm{d}\tilde{z}_3 \wedge \mathrm{d}\tilde{z}_4), \tag{4.9}
$$

which can be dualised in terms of

$$
\hat{A}_1^{12} = \hat{A}_1^{34} = \frac{\mu}{4\sqrt{2}} \left(\tilde{z}_1 \mathrm{d}\tilde{z}_2 - \tilde{z}_2 \mathrm{d}\tilde{z}_1 + \tilde{z}_3 \mathrm{d}\tilde{z}_4 - \tilde{z}_4 \mathrm{d}\tilde{z}_3\right). \tag{4.10}
$$

We interpret the solution (4.1) as the five-dimensional rotating three-charge black hole in a Gödel universe. Although it is not a solution of minimal five-dimensional supergravity,

JHEP11(2003)033

one might expect that the two limits are. To recover minimal supergravity, the scalars must vanish. So either $\beta = 0$ and $Q_1 = Q_5 = Q_k$ (the BMPV black hole), or $Q_1 = Q_5 = Q_k = J = 0$ (the Gödel universe).

Setting $\beta = 0$ gives the five-dimensional rotating three-charge black hole [30, 31]:

$$
\begin{aligned}
\mathrm{d}s_5^2 &= -(H_1 H_5 H_k)^{-\frac{2}{3}} (\mathrm{d}t - \sigma)^2 + (H_1 H_5 H_k)^{\frac{1}{3}} \mathrm{d}s^2(\mathbb{R}^4)\,, \\
A_1 &= \left(H_k^{-1} - 1\right) \mathrm{d}t - H_k^{-1}\sigma\,, \\
B_2 &= -\eta Q_5 \cos^2\theta \mathrm{d}\tilde{\phi}_1 \wedge \mathrm{d}\tilde{\phi}_2 - H_1^{-1} \mathrm{d}t \wedge \sigma\,, \\
C_1 &= \left(H_1^{-1} - 1\right) \mathrm{d}t - H_1^{-1}\sigma\,, \\
e^{2\Phi} &= H_1 H_5^{-1}\,, \qquad e^{2b\psi} = H_1^{-\frac{1}{3}} H_5^{\frac{1}{3}}\,, \qquad e^{2a\chi} = (H_1 H_5)^{-\frac{1}{6}} H^{\frac{1}{3}}\,,
\end{aligned} \tag{4.11}
$$

where

$$
\sigma = \frac{J}{2r^2}\left(\cos^2\theta \,\mathrm{d}\widetilde{\phi}_1 - \eta \sin^2\theta \,\mathrm{d}\widetilde{\phi}_2\right), \qquad H_k = 1 + \frac{Q_k}{r^2}\,, \tag{4.12}
$$

and the BMPV black hole [9] arises from setting the charges $Q_1 = Q_5 = Q_k \equiv Q$. In this equal-charge case the scalars do indeed vanish, and we have $F_{(1)} = F_{(2)} = F$. To recover the equation

$$
\mathrm{d} \star F + F \wedge F = 0\,, \tag{4.13}
$$

of the minimal theory, we then also need

$$
G_3 = \star F\,. \tag{4.14}
$$

However, an explicit calculation for the solution (4.11) gives

$$
G_3 = +\eta \star F\,, \tag{4.15}
$$

so it would seem that only the right ($\eta = +1$) case is a solution of minimal supergravity. In this sense, the minimal theory is the positive charge sector of ours.

On the other hand, setting the black hole charges to zero gives

$$
\begin{aligned}
\mathrm{d}s_5^2 &= -\,(\mathrm{d}t - \sigma)^2 + \mathrm{d}s^2(\mathbb{R}^4)\,, \\
A_1 &= -\sigma\,,
\end{aligned} \tag{4.16}
$$

where now

$$
\sigma = \beta r^2 \left(\cos^2\theta \,\mathrm{d}\widetilde{\phi}_1 + \epsilon \sin^2\theta \,\mathrm{d}\widetilde{\phi}_2\right), \tag{4.17}
$$

but there are further dual one-forms (4.7) or (4.10), which came originally from the RR five-form field strength in ten dimensions. Since $G_3 = F_{(2)} = 0$, and with $F_{(1)} = \mathrm{d}A_1 = F$, we would need

$$
G_3^{ij} = c^{ij} \star F\,, \qquad \sum_{j>i} c^{ij} c^{ij} = 1\,, \tag{4.18}
$$

to get the minimal equation (4.13) as in appendix B.

It is clear that the potentials given in (4.7) do not fall into this category, although those in (4.10) do. The latter will therefore give a solution of the minimal theory, but only with $\epsilon = +1$ in (4.17). The parity of the Kaluza-Klein vector must match that of the dual one-form coming from the self-dual five-form in ten dimensions. (To match with $\epsilon = -1$ in (4.17), we would need to start with an *anti*-self-dual five-form.) Note that this implies $\epsilon = +\eta$, so to get both BMPV and Gödel solutions of the minimal theory we need to make this choice in (3.5), and this leads to a more complicated, θ dependent, metric. Taking $\epsilon = -\eta$ to make the analysis simpler, means that only our BMPV ($\eta = +1$) *or* our Gödel ($\eta = -1$) solution will be relevant to the minimal theory.

A different approach was recently used by Herdeiro to construct this "BMPV Gödel black hole" [18], taking the extremal limit of a Kerr-Newman-Gödel black hole. This, in turn was constructed *via* a Hassan-Sen (HS) transformation [32] of a Kerr-Gödel black hole found in [33]. We can demonstrate that our black hole indeed recovers the BMPV Gödel solution of [18] as follows. We first consider a rescaling of coordinates by defining "natural" time and radial coordinates, $\bar{t}$ and $\bar{r}$, as those in which the asymptotic behaviour of the general solution (4.1) is

$$\mathrm{d}s_5^2 = -\left(\mathrm{d}\bar{t} - \sigma\right)^2 + \mathrm{d}\bar{r}^2 + \bar{r}^2 \mathrm{d}\Omega_3^2 \,. \tag{4.19}$$

This requires the rescaling

$$\bar{t} = \lambda^{-1/3} t \,, \qquad \bar{r} = \lambda^{1/6} r \,, \tag{4.20}$$

with λ as in (3.29). Such rescalings necessitate further rescalings of the dimensionful parameters

$$\bar{Q}_1 = \lambda^{1/3} Q_1 \,, \qquad \bar{Q}_5 = \lambda^{1/3} Q_5 \,, \qquad \bar{Q}_{k\beta} = \lambda^{1/3}\left(\lambda^{-1} Q_{k\beta}\right) , \qquad \bar{\beta} = \lambda^{-2/3}\beta \,. \tag{4.21}$$

The black hole metric (4.1) then takes the form

$$\mathrm{d}s_5^2 = -\mathbb{H}^{-\frac{2}{3}}\left[\mathrm{d}\bar{t} - \left(\bar{\beta}\bar{r}^2 + \bar{\beta}(\bar{Q}_1 + \bar{Q}_5) + \frac{J_\beta}{2\bar{r}^2}\right)\left(\cos^2\theta\, \mathrm{d}\widetilde{\phi}_1 - \eta \sin^2\theta\, \mathrm{d}\widetilde{\phi}_2\right)\right]^2 + \mathbb{H}^{\frac{1}{3}}\, \mathrm{d}\bar{s}^2(\mathbb{R}^4) \,, \tag{4.22}$$

where

$$\mathbb{H} = \left(1 + \frac{\bar{Q}_1}{\bar{r}^2}\right)\left(1 + \frac{\bar{Q}_5}{\bar{r}^2}\right)\left(1 + \frac{\bar{Q}_{k\beta}}{\bar{r}^2}\right) . \tag{4.23}$$

Taking $\bar{Q}_1 = \bar{Q}_5 = \bar{Q}_{k\beta} = \bar{Q}$, gives the BMPV Gödel black hole of [18]. To reproduce this latter precisely, we should identify the parameters of our solution as

$$\bar{\beta} \to 2J \,, \qquad \bar{Q} \to \mu \,, \qquad J_\beta \to 4\mu\omega \,, \tag{4.24}$$

giving a ten-dimensional interpretation of the five-dimensional charges in [18]: the five-dimensional Gödel parameter, J, becomes the ten-dimensional Gödel parameter $\bar{\beta}$; the five-dimensional mass (or charge), μ, becomes the ten-dimensional charge $\bar{Q}$; and the five-dimensional angular momentum, ω, is related to the ten-dimensional brane rotation J_β. It is intriguing that the GV procedure in ten dimensions is equivalent to a HS transformation in six. We should emphasise, however, that our understanding of the ten-dimensional origins of the BMPV Gödel black hole leads naturally to the more general solution (4.1), with $H_{k\beta}$ as in (3.2)-(3.5). For $\epsilon = +\eta$ (the Gödel rotation and black hole rotation generating opposite angular momenta, J_L or J_R), there is θ dependence in the spacetime fields. The further restriction (3.6) on the charges must also be imposed in this case. It would be interesting to understand this restriction as arising from the mixed angular momenta solution in the five-dimensional language, but for now we will concentrate on the properties of the $\epsilon = -\eta$ version of the metric (4.1), and its rescaled form (4.22).

5. Properties of the black hole

We consider the three-charge black hole in the Gödel universe given by the metric (4.22). As noted in [18], for $J^2 < 4Q_1Q_5Q_k$, the coordinate singularity at $\bar{r} = 0$ is a null surface and can be interpreted as a horizon.[5] In the BMPV case ($\bar{Q}_1 = \bar{Q}_5 = \bar{Q}_{k\beta}$), as in [18] we can introduce a Schwarzschild-like coordinate

$$\bar{R}^2 = \bar{r}^2 + \bar{Q}\,, \tag{5.1}$$

in which the metric becomes

$$ds_5^2 = -\left(1 - \frac{\bar{Q}}{\bar{R}^2}\right)^2 (d\bar{t} - \sigma)^2 + \left(1 - \frac{\bar{Q}}{\bar{R}^2}\right)^{-2} d\bar{R}^2 + \bar{R}^2 d\Omega_3^2\,, \tag{5.2}$$

where

$$\sigma = \frac{1}{\bar{R}^2}\left(1 - \frac{\bar{Q}}{\bar{R}^2}\right)^{-1}\left(\bar{\beta}(\bar{R}^4 - \bar{Q}^2) + \frac{J_\beta}{2}\right)\left(\cos^2\theta\, d\widetilde{\phi}_1 - \eta\sin^2\theta\, d\widetilde{\phi}_2\right). \tag{5.3}$$

There is a physical timelike singularity at $\bar{R} = 0$, where the Ricci scalar diverges, and a horizon at $\bar{R}^2 = \bar{Q}$. As we will observe below, the $\bar{\beta}$ dependence drops out of the metric at the horizon.

To calculate its area, consider the induced metric on a $\bar{r}, \bar{t} = \text{const}$ surface,

$$\begin{aligned} ds_{\text{ind}}^2 &= \mathbb{H}^{\frac{1}{3}}\bar{r}^2\left(d\theta^2 + \cos^2\theta\, d\widetilde{\phi}_1^2 + \sin^2\theta\, d\widetilde{\phi}_2^2\right) - \\ &\quad -\mathbb{H}^{-\frac{2}{3}}\left(\bar{\beta}\bar{r}^2 + \bar{\beta}(\bar{Q}_1 + \bar{Q}_2) + \frac{J_\beta}{2\bar{r}^2}\right)^2\left(\cos^2\theta\, d\widetilde{\phi}_1 - \eta\sin^2\theta\, d\widetilde{\phi}_2\right)^2. \end{aligned} \tag{5.4}$$

The determinant at $\bar{r} = 0$ takes the simple form

$$h_{\text{ind}}\Big|_{\bar{r}=0} = \left(\bar{Q}_1\bar{Q}_5\bar{Q}_{k\beta} - \frac{J_\beta^2}{4}\right)\sin^2\theta\cos^2\theta\,, \tag{5.5}$$

giving the horizon area

$$\mathcal{A} = \int_{S^3}\sqrt{h_{\text{ind}}}\Big|_{\bar{r}=0} = 2\pi^2\sqrt{\bar{Q}_1\bar{Q}_5\bar{Q}_{k\beta} - \frac{J_\beta^2}{4}}\,. \tag{5.6}$$

[5] In the equal charge case, this can be shown rigorously by introducing coordinates which cover the horizon [27].

This result is appealing, since it takes the standard form of the horizon area of a three-charge black hole, but in terms of the charges which are all modified by the Gödel parameter, β. In terms of the original charges, however, it turns out that we can equally well write the above horizon area as

$$\mathcal{A} = 2\pi^2\sqrt{Q_1 Q_5 Q_k - \frac{J^2}{4}}\,, \tag{5.7}$$

so this *is* still an example of the conjecture of [17], that such a result should be independent of the Gödel parameter. In either case, there is a bound on the angular momentum which takes the same form as that for the three-charge black hole:

$$J^2 \leq 4Q_1Q_5Q_k \quad \Leftrightarrow \quad J_\beta^2 \leq 4\bar{Q}_1\bar{Q}_5\bar{Q}_{k\beta}\,. \tag{5.8}$$

The characterisation of [11] as under- or over-rotating is equally applicable in the Gödel background, in terms of the modified parameters.

This characterisation is also relevant to the discussion of CTCs, which are present for

$$\bar{f}(\bar{r}) \equiv \bar{r}^4\left(\bar{\beta}\bar{r}^2 + \bar{\beta}(\bar{Q}_1 + \bar{Q}_5) + \frac{J_\beta}{2\bar{r}^2}\right)^2 - \bar{r}^6\mathbb{H} > 0\,. \tag{5.9}$$

The general condition for the existence of CTCs at a radius r in terms of the original parameters is therefore

$$f(r) \equiv \frac{J^2}{4} - (r^2+Q_1)(r^2+Q_5)(r^2+Q_k) + \beta^2 r^4(r^2+Q_1)(r^2+Q_5) > 0\,. \tag{5.10}$$

The rotating three-charge black hole is classified as over-rotating when $J^2 > 4Q_1Q_5Q_k$ [11], whence there are CTCs induced by the rotation of the branes, outside of the horizon itself.[6] Correspondingly, for $J^2 < 4Q_1Q_5Q_k$, the under-rotating case of [11], these CTCs are hidden entirely behind the horizon. Indeed, for $\beta = 0$, the condition for CTCs reduces to that for the three-charge black hole

$$J^2 > 4(r^2+Q_1)(r^2+Q_5)(r^2+Q_k)\,. \tag{5.11}$$

Thus CTCs exist outside the black hole horizon if $J^2 > 4Q_1Q_5Q_k$, but these spacetimes are sick from varying viewpoints [11]–[14]. The rotation induced by the Gödel background does not change these properties, since the deformation vanishes as $r \to 0$. What the Gödel rotation *does* do, however, is give rise to further CTCs at large r. In the pure Gödel case, there are CTCs when $r > 1/\beta$. For r much larger than any length scale associated with the black hole, they persist in the more general system also. (As $\beta \to 0$, these CTCs are pushed off to infinity.)

There are other simple results which may be extracted from (5.10) . For a non-rotating black hole ($J = 0$) in a Gödel universe, the condition reduces to

$$r^2 > \frac{1}{\beta^2} + 2Q_k\,. \tag{5.12}$$

Relative to the Gödel background, the presence of the black hole charge takes the VL surface, which we shall denote by r_*, to a larger radius.

[6] In the $\beta = 0$ case, this corresponds to a breakdown of unitarity in the dual field theory [12].

In order to consider more general results regarding the existence of CTCs in the Gödel black hole, we should study the equation (5.10) without setting any of the parameters to zero. Before doing so, however, we will consider the notion of "holographic protection of chronology". In [4], it was demonstrated that, relative to an observer at $r = 0$, there is a preferred holographic screen in the Gödel universe located at $r = r_{\mathcal{H}} \equiv \sqrt{3}/(2\beta)$. Since $r_{\mathcal{H}} < r_*$, there are no CTCs enclosed within the screen, and in this sense chronology is protected; the holographic screen carves out a causally well-behaved region of the spacetime. It is of interest to investigate if this possible method of protecting chronology can exist in the Gödel black hole metric constructed here. As in [4], the spherical symmetry of the metric allows us to construct the preferred holographic screen with relative ease. It is simply the constant r surface of maximal area (with respect to r). From (5.4), we have

$$\mathcal{A}(r) = \int_{S^3} \sqrt{h_{\text{ind}}} = 2\pi^2 \sqrt{-f(r)}\,. \tag{5.13}$$

Maximising $\mathcal{A}(r)$ involves extracting the roots of

$$\frac{\partial}{\partial r} f(r) = 0\,, \tag{5.14}$$

which is explicitly written as

$$r\left((Q_1 + Q_5 + 2r^2)(Q_k + r^2 - \beta^2 r^4) + (Q_1 + r^2)(Q_5 + r^2)(1 - 2\beta^2 r^2)\right) = 0\,. \tag{5.15}$$

We can observe immediately that the position of any holographic screen is independent of J. Furthermore, there is always a screen at $r = 0$, associated with the black hole horizon. The second screen, which was interpreted as a chronology protecting screen in the Gödel universe [4], only exists when $\beta \neq 0$ as expected. In the case in which we switch off the three black hole charges, Q_1, Q_5 and Q_k, this screen occurs at $r = \sqrt{3}/(2\beta)$, as in [4]. However, the fact that the result is independent of J, whilst J does affect the position of the VL surface, raises the question of whether the screen always lies within the VL surface. We proceed to study this issue by investigating the function $f(r)$.

Since f is a function of r^2 only, we simplify this study by considering f as a quartic function in $x = r^2$:

$$f(x) = \frac{J^2}{4} - (x + Q_1)(x + Q_5)(x + Q_k) + \beta^2 x^2 (x + Q_1)(x + Q_5)\,. \tag{5.16}$$

Note that $x = 0$ represents the horizon of the black hole and the physical values of x are given by the range $x > 0$. Within this range, a positive value of f will imply the existence of CTCs and an extrema of f will represent a holographic screen. The general behaviour of f can be constructed by considering first $f(x;\, J = \beta = 0)$. This function has leading behaviour $f \sim -x^3$ as $|x| \to \infty$, and three roots, each at negative values of x. In particular, we thus know that $f'(0;\, J = \beta = 0) < 0$ and, for $x \geq 0$, $f(x;\, J = \beta = 0)$ is a monotonically decreasing function.

Now consider the effect of the J^2-term on this function, i.e., consider $f(x;\, \beta = 0)$. Given that this additional term is independent of x, its effect is just to translate the previous graph up by some constant amount. The value of $f(x;\beta=0)$ at $x=0$ is determined by the size of J relative to the product of the charges, Q_i, and this determines if CTCs associated with the black hole are present outside of the horizon, i.e., whether we are considering an over- or under-rotating black hole. Consider for now the under-rotating case, for which we recall that $f(0;\beta=0)<0$, $f'(0;\beta=0)<0$ and $f(x;\beta=0)$ is monotonically decreasing for $x\geq 0$.

The final term which we add to this function is the β^2-term which is monotonically increasing in the range of interest, $x\geq 0$. This term has the important properties that $f(0;J=Q_i=0)=0$ and $f'(0;J=Q_i=0)=0$, whilst its leading behaviour, $f\to\beta^2 x^4$ as $x\to\infty$, dominates that of the existing cubic function. Therefore the characteristic behaviour of the complete quartic function, $f(x)$, in the region of interest is always of the form given in figure 1. We note in particular that, despite the presence of the rotating black hole, there always exists a holographic screen, outside of the black hole horizon, which shields the CTCs associated with the Gödel-like asymptotics. Furthermore, the single minimum of the function ensures that the mixing of the Gödel rotation and black hole rotation never generates extra isolated regions of CTCs, other than those which first occur at the VL surface, $r=r_*$, and persist all the way out to infinity — recall, however, that the position of this VL surface *is* dependent on the black hole parameters.

For completeness, consider what occurs in the case of the over-rotating Gödel black hole. The arguments of the previous paragraphs remain, but we note that there is no reason for the CTCs associated with the over-rotating black hole not to mix with those associated with the Gödel universe. Indeed, if the bound on the black hole angular momentum, $J\leq 4Q_1Q_5Q_k$, is violated, then for sufficiently large values of the Gödel parameter, β, the minimum in $f(x)$ will occur in the region where the black hole CTCs still persist — the spacetime outside of the horizon will then be causally sick everywhere. The behaviour of $f(x)$ in this case is illustrated in figure 2.

It is important that we have been able to draw these conclusions only because we have a proper (ten-dimensional) microscopic understanding of the Gödel black hole. Consider, that is, asking similar questions regarding the possibility of holographic protection of chronology in the under-rotating BMPV-Gödel black hole of [18]. This arises from setting $\bar{Q}_1=\bar{Q}_5=\bar{Q}_{k\beta}=\bar{Q}$ in (4.22). In that case, one can follow the line of argument described previously, but now beginning with the function

$$\bar{f}(\bar{x})=\frac{J_\beta^2}{4}-(\bar{x}+\bar{Q})^3+\bar{\beta}(x^2+2\bar{Q}\bar{x})\left(\bar{\beta}(x^2+2\bar{Q}\bar{x})+J_\beta\right). \tag{5.17}$$

The difference arises in the fact that the $\bar{\beta}$-term does not have a vanishing first derivative. It is thus possible that, for a given choice of parameters, the minimum in $\bar{x}>0$ does not occur and one would conclude that holographic protection of chronology fails. This indeed occurs when we pick parameters satisfying

$$16\bar{\beta}^2\left(2\bar{\beta}J_\beta-3\bar{Q}\right)>\left(3-8\bar{\beta}^2\bar{Q}\right)^2, \tag{5.18}$$

a condition which arises by solving explicitly for the roots of $\bar{f}'(\bar{x})=0$.

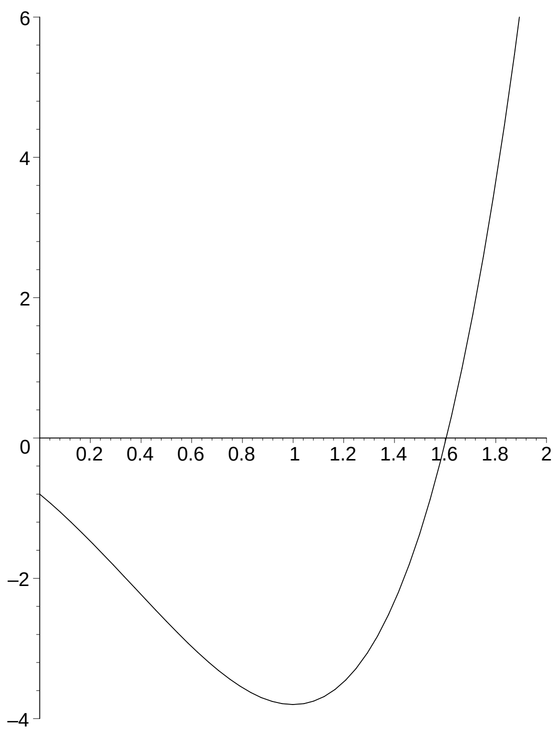

Figure 1: Characteristic behaviour of $f(x)$ for the under-rotating black hole, plotted in the dimensionless variables, $\beta^6 f$ *vs.* $\beta^2 x$. CTCs occur where $f(x) > 0$ and the holographic screen is at the minimum.

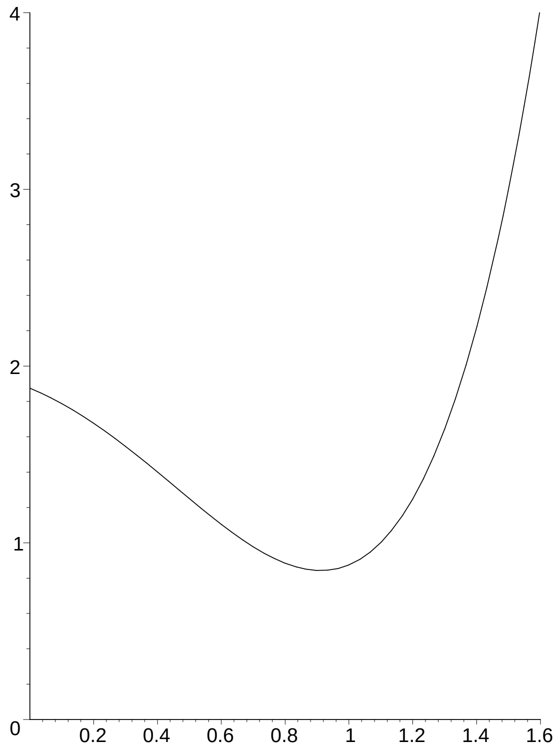

Figure 2: Characteristic behaviour of $f(x)$ for the over-rotating black hole with CTCs everywhere (which arises for sufficiently large β), plotted in the dimensionless variables, $\beta^6 f$ *vs.* $\beta^2 x$.

Without an underlying microscopic knowledge of the parameters, such an inequality can be satisfied. Consider, however, the subsequent requirement that $2\bar{\beta}J_\beta - 3\bar{Q} > 0$. Converting back to the parameters, J, β and Q, this inequality is equivalent to

$$J - \frac{1}{2\beta}\left(3Q + 2\beta^2 Q^2\right) > 0\,. \tag{5.19}$$

Given that, for an under-rotating black hole we have $J^2 < 4Q^3$, we know that this can only be satisfied if

$$16\beta^2 Q^3 - \left(3Q + 2\beta^2 Q^2\right)^2 = -4Q^2\left(\left(\beta^2 Q - \frac{1}{2}\right)^2 + 2\right) > 0\,, \tag{5.20}$$

an obvious impossibility — the parameters are not allowed to be chosen in such a way that would violate the holographic protection of chronology! The phase space of the parameters $\bar{Q}$, $\bar{\beta}$ and J_β is thus reduced by our knowledge of the underlying microscopic description of the BMPV Gödel black hole.

6. Discussion

In this paper we have found a solution corresponding to a rotating, charged black hole in a Gödel universe. Our solution generalises previously known solutions and provides a basis for a microscopic description in terms of D-branes. We hope that it will serve as a laboratory to investigate the issue of CTCs in string theory. In particular, we hope that it can throw light on possible mechanisms that either prevent the formation of CTCs, or else render them harmless.

It has been argued [14], for example, that there is just such a mechanism at work in the case of rotating, charged black holes in an asymptotically flat background. When one attempts to build such a black hole step-by-step, by throwing in matter, it is found that the total angular momentum can never exceed the bound for which CTCs are formed. The creation of a black hole with CTCs is therefore claimed to be impossible [14]. Similar arguments concerning the Gödel universe have been advanced in [16].

For such Gödel universes, it has been argued [4] that there are holographic screens which shield local observers from the effects of any (Gödel) CTC. In the solution that we have found, one can study both types of CTCs simultaneously and it would be very interesting to analyse in what sense the two classes of CTCs differ and in what sense they are similar.

It is by no means obvious that the existence of the holographic screens guarantee that the Gödel CTCs are harmless. In fact, as shown in [15], it seems quite possible for a probe to follow a CTC all the way round — despite the presence of the holographic screens — and it is not at all clear how paradoxes can be avoided. An alternative possibility [16], more in line with [14], is simply to say that a Gödel universe can not be formed in the first place and that the solution therefore is unphysical. In [14] the argument was based on whether local physics could give rise to CTCs in a universe that was well behaved on large scales. In case of the Gödel universe — with CTCs everywhere — it is not clear how a similar argument can be formulated in any interesting way. Clearly much remains to be understood.

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A. Ten-dimensional supergravity theories

A consistent truncation of the bosonic sector of type IIB supergravity is provided by the metric, g_{ab}, the dilaton, Φ, and the Ramond-Ramond (RR) potentials, C_2 and C_4. The RR potentials give rise to the gauge invariant field strengths, $F_3 = \mathrm{d}C_2$ and $F_5 = \mathrm{d}C_4$, the latter being self-dual. One can, however, take the action to be [34]

$$2\kappa_{10}^2\, S_{\mathrm{IIB}} = \int \mathrm{d}^{10}x\sqrt{-g_{10}}\left(R_{10} - \frac{1}{2}\partial\Phi\cdot\partial\Phi\right) - \frac{1}{2}\int\left(e^{\Phi}\star F_3\wedge F_3 + \frac{1}{2}\star F_5\wedge F_5\right), \quad \text{(A.1)}$$

and impose the self-duality condition, $\star F_5 = F_5$, at the level of the equations of motion, which are [35]

$$\begin{aligned}
\mathcal{R}_{ab} &= \frac{1}{2}\partial_a\Phi\partial_b\Phi + \frac{1}{96}F_{ac_1\ldots c_4}F_b{}^{c_1\ldots c_4} + \frac{1}{4}e^{\Phi}\left(F_{acd}F_b{}^{cd} - \frac{1}{12}g_{ab}F_3^2\right),\\
\Box\Phi &= \frac{1}{12}e^{\Phi}F_3^2\,, \qquad \mathrm{d}F_5 = 0\,, \qquad F_5 = \star F_5\,,\\
\mathrm{d}\left(e^{\Phi}\star F_3\right) &= 0\,, \qquad F_5\wedge F_3 = 0\,.
\end{aligned} \tag{A.2}$$

The complete set of T-duality rules to take us to the IIA theory can be found in, e.g., [36]. In our case, T-duality along y_5 gives the type-IIA fields

$$\begin{aligned}
\tilde{g}_{55} &= \frac{1}{g_{55}}\,, \qquad \tilde{g}_{ab} = g_{ab} - \frac{g_{a5}g_{b5}}{g_{55}}\,, \qquad \tilde{B}_{a5} = \frac{g_{a5}}{g_{55}}\,, \qquad e^{2\tilde{\Phi}} = \frac{e^{2\Phi}}{g_{55}}\,,\\
\tilde{C}^{(1)}_a &= C^{(2)}_{5a}\,, \qquad \tilde{C}^{(3)}_{abc} = C^{(4)}_{5abc}\,, \qquad \tilde{C}^{(3)}_{5ab} = C^{(2)}_{ab} - \frac{2}{g_{55}}g_{5[a}C^{(2)}_{|5|b]}\,,\\
\tilde{C}^{(5)}_{5abcd} &= C^{(4)}_{abcd} - \frac{4}{g_{55}}g_{5[a}C^{(4)}_{|5|bcd]}\,,
\end{aligned} \tag{A.3}$$

where a, b run over all directions except y_5. Note that the T-duality here is acting on the string frame metric, whereas everything in the text is written in terms of the Einstein frame metric $g^{\mathrm{E}}_{ab} = e^{-\Phi/2}g^{\mathrm{S}}_{ab}$.

The bosonic sector of type-IIA supergravity is provided by the metric, g_{ab}, the dilaton, Φ, the Neveu Schwarz-Neveu Schwarz (NS-NS) two-form potential, B_2, and the RR potentials, C_1 and C_3. The potentials give rise to a NS-NS field strength, $H_3 = \mathrm{d}B_2$, and the gauge invariant RR field strengths, $F_2 = \mathrm{d}C_1$ and $G_4 = \mathrm{d}C_3 + C_1\wedge H_3$. The action is

$$\begin{aligned}
2\kappa_{10}^2\, S_{\mathrm{IIA}} = &\int \mathrm{d}^{10}x\sqrt{-g}\left(R - \frac{1}{2}\partial\Phi\cdot\partial\Phi\right) - \\
&-\frac{1}{2}\int\left(e^{3\Phi/2}\star F_2\wedge F_2 + e^{\Phi/2}\star G_4\wedge G_4 + e^{-\Phi}\star H_3\wedge H_3 - B_2\wedge\mathrm{d}C_3\wedge\mathrm{d}C_3\right),
\end{aligned} \tag{A.4}$$

with the equations of motion

$$\begin{aligned}
R_{ab} &= \frac{1}{2}\partial_a\Phi\partial_b\Phi + \frac{1}{4}e^{-\Phi}\left(H_{acd}H_b{}^{cd} - \frac{1}{12}g_{ab}H_3^2\right) + \\
&\quad + \frac{1}{2}e^{3\Phi/2}\left(F_{ac}F_b{}^c - \frac{1}{16}g_{ab}F_2^2\right) + \frac{1}{4}e^{\Phi/2}\left(G_{acde}G_b{}^{cde} - \frac{3}{32}g_{ab}G_4^2\right),\\
\Box\Phi &= -\frac{1}{12}e^{-\Phi}H_3^2 + \frac{3}{8}e^{3\Phi/2}F_2^2 + \frac{1}{96}e^{\Phi/2}G_4^2\,,\\
\mathrm{d}\left(e^{3\Phi/2}\star F_2\right) &= e^{\Phi/2}\star G_4\wedge H_3\,, \qquad \mathrm{d}\left(e^{\Phi/2}\star G_4\right) = G_4\wedge H_3\,,\\
\mathrm{d}\left(e^{-\Phi}\star H_3\right) &= -\,e^{\Phi/2}\star G_4\wedge F_2 + \frac{1}{2}G_4\wedge G_4.
\end{aligned} \tag{A.5}$$

In the text, we also have a five-form RR potential, C_5. The associated gauge invariant field strength is dual to G_4:

$$G_6 = F_6 + C_3\wedge H_3 = e^{\Phi/2}\star G_4\,, \qquad F_6 = \mathrm{d}C_5\,, \tag{A.6}$$

JHEP11(2003)033

so the above equation of motion for G_4 becomes a Bianchi identity,

$$\mathrm{d}G_6 = G_4 \wedge H_3 \,, \tag{A.7}$$

for G_6. The Bianchi identity, $\mathrm{d}G_4 = F_2 \wedge H_3$, for G_4 then becomes the equation of motion,

$$\mathrm{d}\left(e^{-\Phi/2} \star G_6\right) = -F_2 \wedge H_3 \,, \tag{A.8}$$

for G_6.

B. Five-dimensional supergravity theories

The relevant reduction of type-IIB supergravity to five dimensions is derived in [27]. We extend this to include the four-form RR potential. Reducing on $T^4 \times S^1$, the ansatz with the five-dimensional metric written in the Einstein frame is

$$\mathrm{d}s_{10}^2 = e^{2a\lambda}\left(e^{2b\psi}\mathrm{d}s_5^2 + e^{-8a\lambda}(\mathrm{d}y_5 + A_1)^2\right) + e^{-3b\psi/2}\mathrm{d}s^2(T^4)\,, \tag{B.1}$$

Replacing λ with[7]

$$\chi(x) = -\frac{1}{4}\frac{b}{a}\psi(x) - \lambda(x)\,, \tag{B.2}$$

and taking $a^2 = 1/24$ and $b^2 = 1/9$, gives rise to canonically normalised five-dimensional scalars. The dilaton reduces trivially and the two-form gauge field as

$$C_2 = B_2 + C_1 \wedge \mathrm{d}y_5\,, \tag{B.3}$$

where B_2 and C_1 are five-dimensional gauge potentials. The modified field strength which appears in the five-dimensional action is

$$G_3 = H_3 - F_2 \wedge A_1\,, \qquad H_3 = \mathrm{d}B_2\,, \qquad F_2 = \mathrm{d}C_1\,. \tag{B.4}$$

For the four-form gauge field we take an ansatz adapted to the specific case considered in the text, writing

$$C_4 = \frac{1}{2}\mathrm{d}y_i \wedge \mathrm{d}y_j \wedge \left(\xi_2^{ij} + \xi_1^{ij} \wedge \mathrm{d}y_5\right). \tag{B.5}$$

The notation $\xi_p^{ij} = -\xi_p^{ji}$ denotes a collection of five-dimensional p-forms. With $F_3^{ij} = \mathrm{d}\xi_2^{ij}$ and $F_2^{ij} = \mathrm{d}\xi_1^{ij}$, the five-form field strength is

$$F_5 = \frac{1}{2}\mathrm{d}y_i \wedge \mathrm{d}y_j \wedge \left(F_3^{ij} + F_2^{ij} \wedge \mathrm{d}y_5\right), \tag{B.6}$$

whereas the natural gauge invariant object is

$$G_3^{ij} = F_3^{ij} - F_2^{ij} \wedge A_1\,. \tag{B.7}$$

Defining

$$\Phi_\pm = 2\left(\pm 2a\chi - \frac{3b}{2}\psi\right), \tag{B.8}$$

[7]This corrects a typographical error in [27].

the five-dimensional action becomes [27]

$$
\begin{aligned}
2\kappa_5^2\, S_5 \,=\, & \int \mathrm{d}^5x\sqrt{-g_5}\left(R_5 - \frac{1}{2}\partial\Phi\cdot\partial\Phi - \frac{1}{2}\partial\psi\cdot\partial\psi - \frac{1}{2}\partial\chi\cdot\partial\chi\right) - \\
& -\frac{1}{2}\int \left(e^{8a\chi}\star F_{(1)}\wedge F_{(1)} + e^{\Phi+\Phi_+}\star G_3\wedge G_3 + e^{\Phi+\Phi_-}\star F_{(2)}\wedge F_{(2)}\right) - \\
& -\frac{1}{2}\int\sum_{j>i}\left(e^{4a\chi}\star G_3^{ij}\wedge G_3^{ij} + e^{-4a\chi}\star F_2^{ij}\wedge F_2^{ij}\right), \qquad \text{(B.9)}
\end{aligned}
$$

where $F_{(1)} = \mathrm{d}A_1$ is the two-form arising from the metric and $F_{(2)} = \mathrm{d}C_1$ is the two-form from the RR field. The Einstein equation is

$$
\begin{aligned}
R_{ab} \,=\, & \frac{1}{2}\left(\partial_a\Phi\partial_b\Phi + \partial_a\psi\partial_b\psi + \partial_a\chi\partial_b\chi\right) + \frac{1}{2}e^{8a\chi}\left(F_{(1)ac}F_{(1)b}{}^{c} - \frac{1}{6}F_{(1)}^2 g_{ab}\right) + \\
& +\frac{1}{4}e^{\Phi+\Phi_+}\left(G_{acd}G_b{}^{cd} - \frac{2}{9}G_3^2 g_{ab}\right) + \frac{1}{2}e^{\Phi+\Phi_-}\left(F_{(2)ac}F_{(2)b}{}^{c} - \frac{1}{6}F_{(2)}^2 g_{ab}\right) + \\
& +\sum_{j>i}\left[\frac{1}{4}e^{4a\chi}\left(G^{ij}_{acd}G_b^{ij\,cd} - \frac{2}{9}(G_3^{ij})^2 g_{ab}\right) + \right. \\
& \qquad \left. + \frac{1}{2}e^{-4a\chi}\left(F^{ij}_{ac}F_b^{ij\,c} - \frac{1}{6}(F_2^{ij})^2 g_{ab}\right)\right], \qquad \text{(B.10)}
\end{aligned}
$$

and the scalar equations are

$$
\Box\Phi \,=\, \frac{1}{12}e^{\Phi+\Phi_+}G_3^2 + \frac{1}{4}e^{\Phi+\Phi_-}F_{(2)}^2 = -\frac{1}{3b}\Box\psi\,, \qquad \text{(B.11)}
$$

$$
\begin{aligned}
\Box\chi \,=\, & \frac{a}{3}\left(e^{\Phi+\Phi_+}G_3^2 + \sum_{j>i}e^{4a\chi}(G_3^{ij})^3\right) - a\left(e^{\Phi+\Phi_-}F_{(2)}^2 + \sum_{j>i}e^{-4a\chi}(F_2^{ij})^2\right) + \\
& +2ae^{8a\chi}F_{(1)}^2\,, \qquad \text{(B.12)}
\end{aligned}
$$

which allow for the solution $\psi = -3b\Phi$. The form field equations are

$$
\mathrm{d}(e^{8a\chi}\star F_{(1)}) \,=\, e^{\Phi+\Phi_+}\star G_3\wedge F_{(2)} + \sum_{j>i}e^{4a\chi}\star G_3^{ij}\wedge F_2^{ij}\,, \qquad \text{(B.13)}
$$

$$
\mathrm{d}(e^{\Phi+\Phi_+}\star G_3) \,=\, 0\,, \qquad \mathrm{d}(e^{4a\chi}\star G_3^{ij}) = 0\,, \qquad \text{(B.14)}
$$

$$
\mathrm{d}\left(e^{\Phi+\Phi_-}\star F_{(2)}\right) \,=\, e^{\Phi+\Phi_+}\star G_3\wedge F_{(1)}\,, \qquad \mathrm{d}\left(e^{-4a\chi}\star F_2^{ij}\right) = e^{4a\chi}\star G_3^{ij}\wedge F_{(1)}\,, \qquad \text{(B.15)}
$$

and the G_3 equations (B.14) can be solved by dualising as

$$
G_3 = e^{-(\Phi+\Phi_+)}\star\mathrm{d}\hat{A}_1 \equiv e^{-(\Phi+\Phi_+)}\star\hat{F}_2\,, \qquad G_3^{ij} = e^{-4a\chi}\star\mathrm{d}\hat{A}_1^{ij} \equiv e^{-4a\chi}\star\hat{F}_2^{ij}\,. \qquad \text{(B.16)}
$$

There are also the non-trivial Bianchi identities

$$
\mathrm{d}G_3 = -F_{(2)}\wedge F_{(1)}\,, \qquad \mathrm{d}G_3^{ij} = -F_2^{ij}\wedge F_{(1)}\,. \qquad \text{(B.17)}
$$

The minimal supergravity theory [37] in five dimensions has no scalars and a single gauge field. Up to rescalings, we recover [27] the equations of motion of that theory,

$$
\mathrm{d}\star F + F\wedge F = 0\,, \qquad \text{(B.18)}
$$

JHEP11(2003)033

by taking $\Phi = \psi = \chi = 0$ in the above and setting all the gauge fields to be equal. Since we will only be interested in obtaining the minimal theory in one of two cases in the text, we first consider $F_2^{ij} = G_3^{ij} = 0$ (relevant to the BMPV black hole) and

$$F_{(1)} = a_1 F\,, \qquad F_{(2)} = a_2 F\,, \qquad G_3 = a_1 a_2 \star F\,, \tag{B.19}$$

where $a_1^2 = a_2^2 = 1$. Then the form equations (B.14) and Bianchi identity (B.17) all give the minimal equation (B.18). We can also consider $F_{(2)} = G_3 = 0$ (relevant to the Gödel universe) and

$$F_{(1)} = aF\,, \qquad F_2^{ij} = c^{ij} F\,, \qquad G_3^{ij} = ac^{ij} \star F\,, \tag{B.20}$$

with

$$a^2 = 1\,, \qquad \sum c^{ij} c^{ij} = 1\,, \tag{B.21}$$

which also gives the minimal equation (B.18).

Both the BMPV and Gödel solutions can be written as

$$\mathrm{d}s^2 = -H^{-2}(\mathrm{d}t - \sigma)^2 + H\mathrm{d}s^2(\mathbb{R}^4)\,, \tag{B.22}$$

$$A = H^{-1}(\mathrm{d}t - \sigma) - \mathrm{d}t\,, \tag{B.23}$$

and we define $J = \mathrm{d}\sigma$. Then the equation (B.18) is solved for harmonic H and

$$\star_4 J = -J\,. \tag{B.24}$$

Now take

$$\mathrm{d}s^2(\mathbb{R}^4) = \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \cos^2\theta \mathrm{d}\phi_1^2 + \sin^2\theta \mathrm{d}\phi_2^2\right), \tag{B.25}$$

and

$$\sigma = f(r)\left(\cos^2\theta \mathrm{d}\phi_1 - \eta \sin^2\theta \mathrm{d}\phi_2\right). \tag{B.26}$$

We have $\star_4 J = -J$ for

$$f(r) = cr^{-2\eta}\,. \tag{B.27}$$

So we have the BMPV black hole for $\eta = +1$ and the Gödel universe for $\eta = -1$. We should note that this does not appear to agree with the usual conventions [3, 12], in which the Gödel universe has non-zero J_R ($\eta = +1$) and the BMPV black hole has non-zero J_L ($\eta = -1$). To agree with this, we would need to reduce the negative charge sector of the IIB theory. Indeed, if we flip the sign of the gauge field in (B.23) above, we can recover the other two solutions (the BMPV black hole for $\eta = -1$ and the Gödel universe for $\eta = +1$). However, we would also need to take an *anti*-self-dual five-form in ten dimensions, which is why we have decided to take the "opposite" conventions to those of [3, 12].

References

[1] K. Gödel, *An example of a new type of cosmological solutions of Einstein's field equations of graviation*, *Rev. Mod. Phys.* **21** (1949) 447.

[2] S.W. Hawking, *The chronology protection conjecture*, *Phys. Rev.* **D 46** (1992) 603.

JHEP11(2003)033

[3] J.P. Gauntlett, J.B. Gutowski, C.M. Hull, S. Pakis and H.S. Reall, *All supersymmetric solutions of minimal supergravity in five dimensions*, *Class. and Quant. Grav.* **20** (2003) 4587 [hep-th/0209114].

[4] E.K. Boyda, S. Ganguli, P. Hořava and U. Varadarajan, *Holographic protection of chronology in universes of the Gödel type*, *Phys. Rev.* **D 67** (2003) 106003 [hep-th/0212087].

[5] T. Harmark and T. Takayanagi, *Supersymmetric Gödel universes in string theory*, *Nucl. Phys.* **B 662** (2003) 3 [hep-th/0301206].

[6] D. Brecher, P.A. DeBoer, D.C. Page and M. Rozali, *Closed timelike curves and holography in compact plane waves*, *J. High Energy Phys.* **10** (2003) 031 [hep-th/0306190].

[7] V.E. Hubeny, M. Rangamani and S.F. Ross, *Causal inheritance in plane wave quotients*, hep-th/0307257.

[8] D. Brace, C.A.R. Herdeiro and S. Hirano, *Classical and quantum strings in compactified PP-waves and Gödel type universes*, hep-th/0307265.

[9] J.C. Breckenridge, R.C. Myers, A.W. Peet and C. Vafa, *D-branes and spinning black holes*, *Phys. Lett.* **B 391** (1997) 93 [hep-th/9602065].

[10] J.P. Gauntlett, R.C. Myers and P.K. Townsend, *Black holes of $D = 5$ supergravity*, *Class. and Quant. Grav.* **16** (1999) 1 [hep-th/9810204].

[11] G.W. Gibbons and C.A.R. Herdeiro, *Supersymmetric rotating black holes and causality violation*, *Class. and Quant. Grav.* **16** (1999) 3619 [hep-th/9906098].

[12] C.A.R. Herdeiro, *Special properties of five dimensional BPS rotating black holes*, *Nucl. Phys.* **B 582** (2000) 363 [hep-th/0003063].

[13] L. Järv and C.V. Johnson, *Rotating black holes, closed time-like curves, thermodynamics and the enhancon mechanism*, *Phys. Rev.* **D 67** (2003) 066003 [hep-th/0211097].

[14] L. Dyson, *Chronology protection in string theory*, hep-th/0302052.

[15] Y. Hikida and S.-J. Rey, *Can branes travel beyond CTC horizon in Gödel universe?*, *Nucl. Phys.* **B 669** (2003) 57 [hep-th/0306148].

[16] N. Drukker, B. Fiol and J. Simon, *Gödel's universe in a supertube shroud*, hep-th/0306057.

[17] E.G. Gimon, A. Hashimoto, V.E. Hubeny, O. Lunin and M. Rangamani, *Black strings in asymptotically plane wave geometries*, *J. High Energy Phys.* **08** (2003) 035 [hep-th/0306131].

[18] C.A.R. Herdeiro, *The Kerr-Newman-Gödel black hole*, *Class. and Quant. Grav.* **20** (2003) 4891 [hep-th/0307194].

[19] P. Bain, P. Meessen and M. Zamaklar, *Supergravity solutions for D-branes in hpp-wave backgrounds*, *Class. and Quant. Grav.* **20** (2003) 913 [hep-th/0205106].

[20] A. Biswas, A. Kumar and K.L. Panigrahi, *p-p' branes in PP-wave background*, *Phys. Rev.* **D 66** (2002) 126002 [hep-th/0208042].

[21] N. Ohta, K.L. Panigrahi and Sanjay, *Intersecting branes in PP-wave spacetime*, *Nucl. Phys.* **B 674** (2003) 306 [hep-th/0306186].

[22] J.T. Liu, L.A. Pando Zayas and D. Vaman, *On horizons and plane waves*, *Class. and Quant. Grav.* **20** (2003) 4343 [hep-th/0301187].

[23] D. Garfinkle, *Traveling waves in strongly gravitating cosmic strings*, *Phys. Rev.* **D 41** (1990) 1112.

[24] D. Garfinkle and T. Vachaspati, *Cosmic string traveling waves*, *Phys. Rev.* **D 42** (1990) 1960.

[25] D. Garfinkle, *Black string traveling waves*, *Phys. Rev.* **D 46** (1992) 4286 [gr-qc/9209002].

[26] V.E. Hubeny and M. Rangamani, *Generating asymptotically plane wave spacetimes*, *J. High Energy Phys.* **01** (2003) 031 [hep-th/0211206].

[27] C.A.R. Herdeiro, *Spinning deformations of the D1-D5 system and a geometric resolution of closed timelike curves*, *Nucl. Phys.* **B 665** (2003) 189 [hep-th/0212002].

[28] D. Brecher, A. Chamblin and H.S. Reall, *AdS/CFT in the infinite momentum frame*, *Nucl. Phys.* **B 607** (2001) 155 [hep-th/0012076].

[29] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *A new maximally supersymmetric background of IIB superstring theory*, *J. High Energy Phys.* **01** (2002) 047 [hep-th/0110242].

[30] M. Cvetič and D. Youm, *General rotating five dimensional black holes of toroidally compactified heterotic string*, *Nucl. Phys.* **B 476** (1996) 118 [hep-th/9603100].

[31] G.T. Horowitz, J.M. Maldacena and A. Strominger, *Nonextremal black hole microstates and U-duality*, *Phys. Lett.* **B 383** (1996) 151 [hep-th/9603109].

[32] S.F. Hassan and A. Sen, *Twisting classical solutions in heterotic string theory*, *Nucl. Phys.* **B 375** (1992) 103 [hep-th/9109038].

[33] E.G. Gimon and A. Hashimoto, *Black holes in Gödel universes and PP-waves*, *Phys. Rev. Lett.* **91** (2003) 021601 [hep-th/0304181].

[34] E. Bergshoeff, H.J. Boonstra and T. Ortin, *S duality and dyonic p-brane solutions in type-II string theory*, *Phys. Rev.* **D 53** (1996) 7206 [hep-th/9508091].

[35] J.H. Schwarz, *Covariant field equations of chiral $N = 2$ $D = 10$ supergravity*, *Nucl. Phys.* **B 226** (1983) 269.

[36] C.V. Johnson, *D-branes*, Cambridge University Press, Cambridge 2003.

[37] E. Cremmer, *Supergravities in 5 dimensions*, in *Superspace and supergravity*, S.W. Hawking and M. Roček eds., CUP, 1981.

JHEP11(2003)033